GAUGES AND THEIR DENSITIES

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1. **Introduction.** The method of constructing measures from gauges in a metric space is particularly important in measure theory. Examples of such include the family of Hausdorff measures in a general metric space, and the integral-geometric (Favard) measures in Euclidean space. The original idea is due to Carathéodory [2]. In generating a measure from a gauge, one considers countable coverings of a given set by elements in the domain of the gauge and only considers the sum of the corresponding values of the gauge. The main object of this paper is to investigate the relationship of the individual terms in the sum to the set that is covered. In particular, one can ask how well the gauge value of a single element of the covering approximates the measure of that part of the covered set within the given element. This leads naturally to the definition of upper and lower gauge densities to be found in §3. In this paper, results are only obtained for upper densities.

In §3, under rather general conditions, it is proved that the upper gauge density is not less than unity for almost all points of a given set (Theorems 3.2, 3.3, 3.4). Here as in the remainder of the paper, measurability of the set is not assumed.

In §4, more conditions are imposed on both the gauge and the metric space, leading to the definition of diametric gauges, However, the concept is still general enough to include sphere and Hausdorff measure in Euclidean space or in sufficiently well-behaved manifolds. For diametric gauges and for sets of finite (outer) measure, it is proved that the upper density is unity almost everywhere in the set (Theorem 4.4). Theorem 4.2 appeared in the special case of Hausdorff measure on the real line in a paper of Besicovitch and Moran [1]. Finally, Theorem 4.5 states that for diametric gauges and sets of finite measure, one can use the gauge to approximate simultaneously in the large and in the small; thus one can cover with sets, for most of which the gauge value approximates the measure covered, and such that the sum of the gauge values approximates the measure of the whole set.

A good summary of earlier and related results can be found in Federer [3, especially §3].

2. **Preliminaries.** Throughout this paper, the following will be assumed:

$\varnothing$ denotes the null set.

$\infty/a = \infty$ if $a > 0$.

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If (R, ρ) is a metric space, $x \in R$, $0 < r < \infty$, then $U(x, r)$ will denote $\{y \in R \mid \rho(x,y) < r\}$. Also, for $A \subset R$, $\operatorname{diam} A$ will denote the diameter of the set A.

DEFINITION 2.1. Let (R,ρ) be a metric space. We shall say that g is a gauge over R if g is a nonnegative real-valued function defined over a class M of subsets of R, $\varnothing \in M$ and $g(\varnothing) = 0$. We shall say that the (outer) measure m is generated by the gauge g if for all $A \subset R$,

$$m(A) = \lim_{r \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} g(M_i) \,\middle|\, A \subset \bigcup_{i=1}^{\infty} M_i,\ \operatorname{diam} M_i < r,\ M_i \in M \right\}.$$

As usual, a set $A \subset R$ is said to be m-measurable if for all $B \subset R$,

$$m(B) = m(B \cap A) + m(B - A).$$

LEMMA 2.2. Let (R,ρ) be a metric space, g be a gauge over R with domain M, m the outer measure generated by g, $A \subset R$, $m(A) < \infty$. Then for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left(M_0 = \bigcup_{i=1}^{\infty} M_i,\ M_i \in M,\ M_0 \text{ is } m\text{-measurable, } \operatorname{diam} M_i < \delta\right)$$
$$\Rightarrow \left(\sum_{i=1}^{\infty} g(M_i) > m(A \cap M_0) - \varepsilon\right).$$

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ so that

$$\left(A \subset \bigcup_{i=1}^{\infty} N_i,\ \operatorname{diam} N_i < \delta, N_i \in M\right) \Rightarrow \left(\sum_{i=1}^{\infty} g(N_i) > m(A) - \frac{\varepsilon}{2}\right).$$

Let $M_i \in M$, $\operatorname{diam} M_i < \delta$, $M_0 = \bigcup_{i=1}^{\infty} M_i$ be m-measurable. Then $m(A) = m(A \cap M_0) + m(A - M_0)$. Choose $P_i \in M$ so that

$$\operatorname{diam} P_i < \delta, \quad A - M_0 \subset \bigcup_{i=1}^{\infty} P_i,\ \sum_{i=1}^{\infty} g(P_i) < m(A - M_0) + \frac{\varepsilon}{2}.$$

Then since $\{M_i\} \cup \{P_i\}$ covers A and each set in the covering has diameter less than δ, it follows that

$$\begin{aligned}\sum_{i=1}^{\infty} g(M_i) + \sum_{i=1}^{\infty} g(P_i) &> m(A) - \frac{\varepsilon}{2} \\ &= m(A \cap M_0) + m(A - M_0) - \frac{\varepsilon}{2} \\ &> m(A \cap M_0) + \sum_{i=1}^{\infty} g(P_i) - \varepsilon,\end{aligned}$$

$$\sum_{i=1}^{\infty} g(M_i) > m(A \cap M_0) - \varepsilon.$$

REMARK. In Lemma 2.2, the condition that M_0 be m-measurable is usually superfluous in application. This is because the outer measure m has the property of being additive on two sets a positive distance apart, so that Borel sets are m-measurable. In many applications, M consists of Borel sets only.

3. Gauge densities.

Definition 3.1. Let g be a gauge over R with domain M, and let m be the generated outer measure. For $x \in R$, $A \subset R$, we define the upper and lower gauge densities of A at x by

$$D^*g_A(x) = \lim_{n\to\infty} \sup_{N \in M_n} \frac{m(A \cap N)}{g(N)},$$

$$D^*g_A(x) = \lim_{n\to\infty} \inf_{N \in M_n} \frac{m(A \cap N)}{g(N)},$$

where $M_n = \{N \in M \mid x \in N,\ \operatorname{diam} N \leqq 1/n,\ g(N) \neq 0\}$. If $D^*g_A(x) = D_*g_A(x)$, we define $Dg_A(x)$ to be their common value.

Theorem 3.2. *Let g be a gauge over R such that every element of M is m-measurable, $A \subset R$, $m(A) < \infty$. Then $D^*g_A(x) \geqq 1$ for all points of A except for a set of m-measure zero.*

Proof. Let $1 > \varepsilon > 0$ and define $\varepsilon_n = \varepsilon^2/2^{2n+4}$ for each positive integer n. By Lemma 2.2, choose $0 < \delta_n < 1/n$ so that if $M_0 = \bigcup_{i=1}^{\infty} M_i$, $M_i \in M$, $\operatorname{diam} M_i < \delta_n$, then $\sum_{i=1}^{\infty} g(M_i) > m(A \cap M_0) - \varepsilon_n$. Next choose a sequence of sets $N_i \in M$ such that $A \subset \bigcup_{i=1}^{\infty} N_i$, $\operatorname{diam} N_i < \delta_n$, and $\sum_{i=1}^{\infty} g(N_i) < m(A) + \varepsilon_n$.

Relabel those N_i such that $g(N_i) > (1 + \varepsilon/2^n)m(A \cap N_i) > 0$ by the symbols Q_j; relabel those N_i such that $g(N_i) = 0$ by the symbols R_j; relabel those N_i such that $g(N_i) > 0 = m(A \cap N_i)$ by the symbols T_j; and relabel the remaining N_i by the symbols V_j. For each Q_j choose a sequence of sets $W_{j,k} \in M$ such that $A \cap Q_j \subset \bigcup_k W_{j,k}$, $\operatorname{diam} W_{j,k} < \delta_n$ and $\sum_k g(W_{j,k}) < (1 + \varepsilon/2^{n+1}) \cdot m(A \cap Q_j)$. Since $\{W_{j,k}\} \cup \{R_j\} \cup \{T_j\} \cup \{V_j\}$ cover A by sets of diameter less than δ_n,

$$\sum_{j,k} g(W_{j,k}) + \sum_j g(R_j) + \sum_j g(T_j) + \sum_j g(V_j) > m(A) - \varepsilon_n.$$

Also

$$\sum_j g(Q_j) + \sum_j g(R_j) + \sum_j g(T_j) + \sum_j g(V_j) < m(A) + \varepsilon_n.$$

Hence

$$2\varepsilon_n > \sum_j g(Q_j) - \sum_{j,k} g(W_{j,k}) = \sum_j \left(g(Q_j) - \sum_k g(W_{j,k}) \right)$$

$$> \sum_j \left(1 + \frac{\varepsilon}{2^n} - 1 - \frac{\varepsilon}{2^{n+1}} \right) \cdot m(A \cap Q_j) = \frac{\varepsilon}{2^{n+1}} \sum_j m(A \cap Q_j),$$

so that

$$\sum_j m(A \cap Q_j) < \frac{\varepsilon}{2^{n+2}}.$$

An application of Lemma 2.2 shows that

$$m\left(A\cap\bigcup_j R_j\right) < \sum_j g(R_j) + \varepsilon_n = \varepsilon_n < \frac{\varepsilon}{2^{n+2}}.$$

Also

$$0 \leqq m\left(A\cap\bigcup_j T_j\right) \leqq \sum_j m(A\cap T_j) = 0.$$

Hence if we define $B_n = \bigcup_j V_j$, then since $A - B_n$ is covered by the sets $\{Q_j\}$, $\{R_j\}$, $\{T_j\}$, it follows that

$$m(A - B_n) \leqq \sum_j m(A\cap Q_j) + m\left(A\cap\bigcup_j R_j\right) + m\left(A\cap\bigcup_j T_j\right) < \frac{\varepsilon}{2^{n+1}}.$$

Also, if $x \in B_n$, then $x \in V_j$ for some j, and $g(V_j) \neq 0$, $m(A\cap V_j)/g(V_j) \geqq (1+\varepsilon/2^n)^{-1}$. If we next define $B = \bigcap_{n=1}^{\infty} B_n$, then $m(A-B) \leqq \sum_{n=1}^{\infty} m(A - B_n) < \varepsilon$. It is obvious that if $x \in B$, then $D^* g_A(x) \geqq 1$. Since ε was arbitrary, the proof is complete.

THEOREM 3.3. *Let g be a gauge over R such that if $\varnothing \neq N \in M$, then $g(N) > 0$. If $A \subset R$, $m(A) < \infty$, then $D^* g_A(x) \geqq 1$ for all points of A except for a set of m-measure zero.*

Proof. Let $1 > \varepsilon > 0$ and define $\varepsilon_n = \varepsilon^2/2^{2n+4}$ for each positive integer n. From the definition of $m(A)$, choose $0 < \delta_n < 1/n$ so that if $M_i \in M$, $A \subset \bigcup_{i=1}^{\infty} M_i$, $\operatorname{diam} M_i < \delta_n$, then $\sum_{i=1}^{\infty} g(M_i) > m(A) - \varepsilon_n$. Next choose a sequence of sets $N_i \in M$ such that $A \subset \bigcap_{i=1}^{\infty} N_i$, $\operatorname{diam} N_i < \delta_n$ and $\sum_{i=1}^{\infty} g(N_i) < m(A) + \varepsilon_n$. The proof of Theorem 3.2 now carries over if one observes that the class $\{R_j\}$ is either empty or consists of $\varnothing$ alone and hence no application of Lemma 2.2 is required

THEOREM 3.4. *Let*

(1) *g be a gauge over R,*

(2) *$A \subset R$ be arbitrary,*

(3) *R be separable,*

(4) *there exist a $\delta > 0$ such that if $x \in R$, $0 < r < \delta$, then $U(x,r) \in M$ and $g(U(x,r)) > 0$,*

(5) *every element of M be m-measurable.*

Then $D^* g_A(x) \geqq 1$ for all points of A except for a set of m-measure zero.

Proof. We first note that because of condition (5), every set is contained in an m-measurable set of the same measure. Hence for any sequence of sets A_i, $m(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} m(\bigcup_{i=1}^{n} A_i)$.

Let Y be a countable set of points dense in R. Let $B = \{x \in A \mid D^* g_A(x) < 1\}$ and assume $m(B) > 0$. For each $x \in B$, choose a sphere $U_x = U(y,r) \in M$ with $y \in Y$, $x \in U_x$, r a positive rational number, and $m(U_x \cap A) < g(U_x)$. Since the collection $\{U_x \mid x \in B\}$ is countable, order them as $U_1, U_2, \cdots$. Choose the integer

n so that $m(B \cap \bigcup_{i=1}^{n} U_i) > 0$. Let $A_1 = A \cap \bigcup_{i=1}^{n} U_i$, $B_1 = B \cap \bigcup_{i=1}^{n} U_i$. Then $B_1 \subset A_1$, $m(B_1) > 0$, $m(A_1) < \infty$ and for $x \in B_1$, $D^* g_A(x) \leqq D^* g_{A1}(x) < 1$. Since Theorem 3.2 is contradicted, the proof is complete.

REMARK. In Theorems 3.2 and 3.4, the condition that every element of M be m-measurable is usually satisfied in applications, as has been already remarked. We now give an example of a gauge g such that no nonnull element of M is m-measurable. If M_1 is the set of open intervals in Euclidean 1-space E^1, and $g_1(I) = \operatorname{diam} I$ for an open interval I, then the generated measure is Lebesgue 1-dimensional measure, L_1. Let M be the set of all subsets of E^1 of the form $\varnothing$ or $(a,b) \cup C$, where C is L_1 nonmeasurable, $C \subset (b, b+(b-a)^2)$, $a < b$. Again let $g(N) = \operatorname{diam} N$ for $N \in M$. Then g generates L_1 and no nonnull member of M is L_1-measurable.

4. **Diametric gauges.** We now fix the metric space (R, ρ) and the gauge g. Throughout the remainder of this paper, we shall make certain assumptions on the gauge and the metric space which we formalize in the following definition.

DEFINITION 4.1. g is a diametric gauge over R with domain M if

(i) g is a gauge over R with m the generated outer measure,

(ii) there exists a $\delta > 0$ such that if $x \in R$, $0 < r \leqq \delta$, then $U(x,r) \in M$ and $\operatorname{diam} U(x,r) > r$.

(iii) every member of M is m-measurable,

(iv) there exist nondecreasing functions f_1 and f_2 on the nonnegative real numbers and a positive constant k such that $f_1(2r) \leqq k \cdot f_2(r)$ and if $N \in M$, $\operatorname{diam} N = r$, then $f_2(r) \leqq g(N) \leqq f_1(r)$,

(v) there exists a positive integer L such that any open sphere of radius $6r$ can be covered by L open spheres of radius r (L independent of r).

REMARK. In Definition 4.1 (iv) one can replace $2 \cdot r$ by $p \cdot r$ where p is any fixed number greater than one, and similarly in 4.1(v) the radius $6 \cdot r$ can be replaced by $p \cdot r$. It is obvious that the definition is then unchanged. In what follows, condition 4.1(v) can be weakened so that it holds only for r sufficiently small.

THEOREM 4.2. *Let g be a diametric gauge over R, $A \subset R$, $m(A) < \infty$. For every $p > 1$, $\varepsilon > 0$, there exists a $\delta > 0$ and sets C and $B = A - C$ such that*:

(i) *C is the union of elements of M,*

(ii) $m(A \cap C) \leqq \varepsilon$, $m(B) \geqq m(A) - \varepsilon$,

(iii) *if $x \notin C$, $x \in N \in M$, $\operatorname{diam} M \leqq \delta$, then $m(A \cap N) \leqq p \cdot g(N)$,*

(iv) *for all $N \in M$ with $\operatorname{diam} N \leqq \delta$, $m(B \cap N) \leqq p \cdot g(N)$.*

Proof. We exclude the trivial case $m(A) = 0$. Let L be an integer such that for all $0 < r < \infty$, an arbitrary sphere in R of radius $6r$ can be covered by L open spheres of radius r. Choose a constant $k > 1$ such that $f_1(2r) \leqq k \cdot f_2(r)$, where f_1 and f_2 are the functions of 4.1(iv).

Let $u = \varepsilon(p-1)(pkL)^{-1}$, and apply Lemma 2.2 to choose $\delta > 0$ so that $U(x,r) \in M$, diam $U(x,r) > r$ *for* $x \in R$, $0 < r \leqq \delta$, and

$$\left(M_0 = \bigcup_{i=1}^{\infty} M_i,\ M_i \in M,\ \operatorname{diam} M_i \leqq \delta\right) \Rightarrow \left(\sum_{i=1}^{\infty} g(M_i) > m(A \cap M_0) - u\right).$$

Next, define $C = \{x \in R \mid \text{for some } N \in M,\ x \in N,\ \operatorname{diam} N \leqq \delta,\ m(A \cap N) \geqq p \cdot g(N)\}$, and for each nonnegative integer t, define

$$D_t = \left\{N \in M \,\middle|\, \frac{\delta}{2^{t+1}} < \operatorname{diam} N \leqq \frac{\delta}{2^t},\ m(A \cap N) \geqq p \cdot g(N)\right\},$$

$$D = \bigcup_{t=0}^{\infty} D_t,$$

so that $C = \bigcup_{N \in D} N$.

For each $N \in D_t$, we associate a sphere $V(N)$ by choosing $x \in N$ and taking $V(N) = U(x, 3\delta/2^t)$.

If $D_0 \neq \emptyset$, choose $N_{0,1} \in D_0$ so that

$$m(A \cap N_{0,1}) > \frac{1}{k} \sup_{N \in D_0} m(A \cap N).$$

Then $V(N_{0,1})$ can be covered by L open spheres of radius $\delta/2$; call them $V_1, \cdots, V_L$. If $V_i \in D_0$, then $m(A \cap V_i) < k \cdot m(A \cap N_{0,1})$. If $V_i \notin D_0$, then since $\operatorname{diam} V_i < 2 \cdot \operatorname{diam} N_{0,1}$, it follows that $m(A \cap V_i) < p \cdot g(V_i) \leqq p \cdot kg(N_{0,1}) \leqq k \cdot m(A \cap N_{0,1})$. Hence

$$m(A \cap V(N_{0,1})) \leqq \sum_{i=1}^{L} m(A \cap V_i) \leqq k \cdot L \cdot m(A \cap N_{0,1}).$$

Let $D_{0,1} = \{N \in D_0 \mid N \cap N_{0,1} = \emptyset\}$. If $D_{0,1} = \emptyset$, the procedure stops. If $D_{0,1} \neq \emptyset$, choose $N_{0,2} \in D_{0,1}$ so that $m(A \cap N_{0,2}) > (1/k)\sup_{N \in D_{0,1}} m(A \cap N)$. Then $V(N_{0,2})$ can be covered by L open spheres of radius $\delta/2$; call them $W_1, \cdots, W_L$.

If $W_i \cap N_{0,1} \neq \emptyset$, then $W_i \subset V(N_{0,1})$. If $W_i \cap N_{0,1} = \emptyset$ and $W_i \notin D_0$, then

$$m(A \cap W_i) < p \cdot g(W_i) \leqq p \cdot k \cdot g(N_{0,2}) \leqq k \cdot m(A \cap N_{0,2}).$$

If $W_i \in D_{0,1}$, then $m(A \cap W_i) < k \cdot m(A \cap N_{0,2})$. Hence

$$m(A \cap (V(N_{0,2}) - V(N_{0,1}))) \leqq \sum_{W_i, N_{01} = \emptyset} m(A \cap W_i)) \leqq kLm(A \cap N_{0\ 2}),$$

so that

$$\begin{aligned} m(A \cap (V(N_{0,1}) \cup V(N_{0,2}))) &\leqq m(A \cap V(N_{0,1})) \\ &\quad + m(A \cap (V(N_{0,2}) - V(N_{0,1}))) \\ &\leqq k \cdot L(m(A \cap N_{0,1}) + m(A \cap N_{0,2})) \\ &= k \cdot L \cdot m(A \cap (N_{0,1} \cup N_{0,2})). \end{aligned}$$

Since $m(A) < \infty$, the above procedure can be continued only a finite number of times, say $q(0)$ (with $q(0)$ possibly 0), to obtain $N_{0,1}, N_{0,2}, \cdots, N_{0,q(0)}$ with the properties:

(i) $N_{0,i} \in D_0$ for $1 \leqq i \leqq q(0)$,

(ii) $N_{0,i} \cap N_{0,j} = \varnothing$ for $i \neq j$,

(iii) $m(A \cap \bigcup_{i=1}^{q(0)} V(N_{0,i})) \leqq k \cdot L \cdot m(A \cap \bigcup_{i=1}^{q(0)} N_{0,i})$,

(iv) $\bigcup_{N \in D_0} N \subset \bigcup_{i=1}^{q(0)} V(N_{0,i})$.

The last condition is explained by the fact that when the procedure cannot be continued, an arbitrary $N \in D_0$ must intersect some $N_{0,i}$ and then $N \subset V(N_{0,i})$.

Next, let

$$D_{1,0} = \left\{ N \in D_1 \,\middle|\, N \cap \bigcup_{i=1}^{q(0)} N_{0,i} = \varnothing \right\}.$$

We shall only consider the nontrivial case when $D_{1,0} \neq \varnothing$. Choose $N_{1,1} \in D_{1,0}$ so that $m(A \cap N_{1,1}) > (1/k)\sup_{N \in D1,0} m(A \cap N)$. Then $V(N_{1,1})$ can be covered by L spheres of radius $\delta/4$, to be denoted by $Z_1, Z_2, \cdots, Z_L$. If $Z_i \cap \bigcup_{i=1}^{q(0)} N_{0,i} \neq \varnothing$, then $Z_i \subset \bigcup_{i=1}^{q(0)} V(N_{0,i})$. If $Z_i \cap \bigcup_{i=1}^{q(0)} N_{0,i} = \varnothing$, $Z_i \notin D_{1,0}$, then $Z_i \notin D_1$, so that $m(A \cap Z_i) < p \cdot g(Z_i) \leqq p \cdot k \cdot g(N_{1,1}) \leqq km(A \cap N_{1,1})$. If $Z_i \in D_{1,0}$, then $m(A \cap Z_i) < km(A \cap N_{1,1})$. Hence

$$m\left(A \cap \left(V(N_{1,1}) - \bigcup_{i=1}^{q(0)} V(N_{0,i})\right)\right) \leqq kLm(A \cap N_{1,1}),$$

$$m\left(A \cap \left(V(N_{1,1}) \cup \bigcup_{i=1}^{q(0)} V(N_{0,i})\right)\right) \leqq kLm\left(A \cap \left(N_{1,1} \cup \bigcup_{i=1}^{q(0)} N_{0,i}\right)\right).$$

The above procedure can be applied only a finite number of times to the class D_1, and a similar procedure can be carried on to all D_i to finally obtain a collection of sets $N_{i,j}$ ($i \geqq 0$, $i \leqq j \leqq q(i)$) with the properties:

(i) $N_{i,j} \in D_i$ for $i \geqq 0$, $1 \leqq j \leqq q(i)$,

(ii) $N_{i,j} \cap N_{r,s} = \varnothing$ if either $i \neq r$ or $j \neq s$,

(iii) $m(A \cap \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{q(i)} V(N_{i,j})) \leqq kL \cdot m(A \cap \bigcup_{=0}^{\infty} \bigcup_{j=1}^{q(i)} N_{i,j})$,

(iv) $C = \bigcup_{N \in D} N \subset \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{(} V(N_{i,j})$.

By the choice of δ and the fact that $\operatorname{diam} N_{i,j} \leqq \delta$ for all i and j, it follows that

$$\sum_{i=0}^{\infty} \sum_{j=1}^{q(i)} m(A \cap N_{i,j}) = m\left(A \cap \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{q(i)} N_{i,j}\right) < \sum_{i=0}^{\infty} \sum_{j=1}^{q(i)} g(N_{i,j}) + u$$

$$\leqq \frac{1}{p} \sum_{i=0}^{\infty} \sum_{j=1}^{q(i)} m(A \cap N_{i,j}) + u,$$

so that

$$\sum_{i=0}^{\infty} \sum_{j=1}^{q(i)} m(A \cap N_{i,j}) \leqq \frac{pu}{p-1}.$$

Hence

$$\begin{aligned} m(A\cap C) &\leqq m\left(A\cap \bigcup_{i=0}^{\infty}\bigcup_{j=1}^{q(i)} V(N_{i,j})\right) \\ &\leqq k\cdot L\cdot m\left(A\cap\bigcup_{i=0}^{\infty}\bigcup_{j=1}^{q(i)} N_{i,j}\right) \\ &\leqq k\cdot L\cdot \sum_{i=0}^{\infty}\sum_{j=1}^{q(i)} m(A\cap N_{i,j}) \\ &\leqq \frac{p\cdot k\cdot L\cdot u}{p-1} = \varepsilon. \end{aligned}$$

Let $B = A - C$. Then

$$m(A) \leqq m(A - C) + m(A\cap C) \leqq m(B) + \varepsilon.$$

Also, from the definition of C, it follows that if $x \notin C$, $x \in N \in M$, $\operatorname{diam} N \leqq \delta$, then $m(A\cap N) < p\cdot g(N)$. Finally, if $N \in M$, $\operatorname{diam} N \leqq \delta$ and $m(B\cap N) > p\cdot g(N)$, then $m(A\cap N) > p\cdot g(N)$. Hence $N\in D$ and therefore $N\subset C$, $N\cap B = \emptyset$, contradicting $m(N\cap B) > 0$. The proof is complete.

COROLLARY 4.3. *Let g be a diametric gauge over R, $A\subset R$, $m(A) < \infty$. For $\varepsilon > 0$ there exist sets C and $B = A - C$ such that*

(i) *C is the union of elements of M,*

(ii) *$m(A\cap C) \leqq \varepsilon$, $m(B) \geqq m(A) - \varepsilon$,*

(iii) *for each $p > 1$ there is a $\delta > 0$ such that if $x\notin C$, $x\in N\in M$, $\operatorname{diam} N \leqq \delta$, then $m(A\cap N) \leqq p\cdot g(N)$,*

(iv) *for each $p > 1$, there is a $\delta > 0$ such that $m(B\cap N) \leqq p\cdot g(N)$ for all $N\in M$ with $\operatorname{diam} N \leqq \delta$.*

Proof. For $p_j = 1 + 1/j$, use Theorem 4.2 to choose C_j, a union of elements of M, and $\delta_j > 0$ such that:

(i) $m(A\cap C_j) \leqq \varepsilon/2^j$,

(ii) if $x\notin C_j$, $x\in N\in M$, $\operatorname{diam} N \leqq \delta_j$, then $m(A\cap N) \leqq p_j\cdot g(N)$,

(iii) $m((A - C_j)\cap N) \leqq p_j\cdot g(N)$ for all $N\in M$ with $\operatorname{diam} N \leqq \delta_j$.

Define $C = \bigcup_{j=1}^{\infty} C_j$ and $B = A - C$. The remaining details of the proof are easily checked.

REMARK. In a sense, Theorem 4.2 and Corollary 4.3 represent the best possible result. For let R be the Euclidean plane, M the set of all subsets of the plane, $g(N) = \operatorname{diam} N$ for $N\in M$, so that g generates Hausdorff linear measure. Let

$$\begin{aligned} A = \{(x,y)\,|\,x^2+y^2=1\} &\cup \{(x,y)\,|\,x=0,\ -1\leqq y\leqq 1\} \\ &\cup\{(x,y)\,|\,y=0,\ -1\leqq x\leqq 1\}. \end{aligned}$$

Then for the conclusions of either the theorem or corollary to hold, a set of positive measure must be taken away from A.

THEOREM 4.4. *Let g be a diametric gauge over R, $A \subset R$, $m(A) < \infty$. Then $D^* g_A(x) = 1$ for all points of A except for a set of m-measure zero.*

Proof. Apply Theorem 3.2 and Corollary 4.3.

THEOREM 4.5. *Let g be a diametric gauge over R, $A \subset R$, $m(A) < \infty$. For each $\varepsilon > 0$, $p > 1$, there exist two sequences of sets $\{N_i\} \subset M$, $\{Q_i\} \subset M$ such that:*

(i) $A \subset \bigcup_i N_i \cup \bigcup_i Q_i$,

(ii) $\operatorname{diam} N_i \leqq \varepsilon$, $\operatorname{diam} Q_i \leqq \varepsilon$ *for each i,*

(iii) $m(A) - \varepsilon \leqq \sum_i g(N_i) + \sum_i g(Q_i) \leqq m(A) + \varepsilon$,

(iv) $\sum_i g(Q_i) \leqq \varepsilon$,

(v) $p^{-1} \cdot g(N_i) \leqq m(A \cap N_i) \leqq p g(N_i)$ *for each i.*

Proof. We exclude the trivial case $m(A) = 0$. Without loss of generality, we assume $(p-1)\varepsilon < m(A)$ and $p < 2$. Apply Theorem 4.2 and Lemma 2.2 to choose a set C and a δ, $0 < \delta < \varepsilon$, such that:

(i) $(M_i \in M,\ \operatorname{diam} M_i \leqq \delta) \Rightarrow (\sum_{i=1}^{\infty} g(M_i) \geqq m(A \cap \bigcup_{i=1}^{\infty} M_i) - \varepsilon/4)$,

(ii) $m(A \cap C) < \varepsilon/4$,

(iii) $$(x \notin C,\ x \in N \in M,\ \operatorname{diam} N \leqq \delta) \Rightarrow \begin{cases} m(A \cap N) \leqq \left(1 - \dfrac{(p-1)\varepsilon}{16 m(A)}\right)^{-1} \cdot g(N), \\ m(A \cap N) \leqq p \cdot g(N). \end{cases}$$

Let $B = A - C$. Choose a sequence $\{T_i\}$ such that each $T_i \in M$, $\operatorname{diam} T_i \leqq \delta$, $T_i \cap B \neq \varnothing$, $B \subset \bigcup_i T_i$, $\sum_i g(T_i) \leqq m(B) + (p-1)\varepsilon/16$. Denote by $\{V_i\}$ those T_i for which $g(T_i) > p \cdot m(B \cap T_i)$ and denote the remaining T_i by $\{N_i\}$. Then $q = (1 - (p-1)\varepsilon/16 m(A))^{-1}$,

$$\begin{aligned} m(B) + \frac{(p-1)\varepsilon}{16} &\geqq \sum_i g(T_i) = \sum_i g(V_i) + \sum_i g(N_i) \\ &\geqq p \sum_i m(B \cap V_i) + \frac{1}{q} \sum_i m(A \cap N_i) \\ &\geqq (p-1) \sum_i m(B \cap V_i) + \frac{1}{q} m(B), \end{aligned}$$

so that

$$\sum_i m(B \cap V_i) \leqq \left(\left(1 - \frac{1}{q}\right) m(B) + \frac{(p-1)\varepsilon}{16}\right)(p-1)^{-1} \leqq \frac{\varepsilon}{8},$$

$$m\left(B \cap \bigcup_i V_i\right) \leqq \frac{\varepsilon}{8}.$$

Hence

$$\sum_i g(N_i) \geqq m\left(A \cap \bigcup_i N_i\right) - \frac{\varepsilon}{4} \geqq m\left(B - \bigcup_i V_i\right) - \frac{\varepsilon}{4}$$

$$\geqq m(B) - m\left(B \cap \bigcup_i V_i\right) - \frac{\varepsilon}{4} \geqq m(B) - \frac{3\varepsilon}{8},$$

$$\sum g(V_i) = \sum_i g(T_i) - \sum_i g(N_i) \leqq m(B)$$

$$+ \frac{\varepsilon}{16} - \left(m(B) - \frac{3\varepsilon}{8}\right) < \frac{\varepsilon}{2}.$$

Finally, choose a sequence $\{R_i\}$ such that for each i, $R_i \in M$, diam $R_i \leqq \delta$, $A \cap C \subset \bigcup_i R_i$, $\Sigma_i g(R_i) \leqq \varepsilon/2$. The proof is completed by denoting $\{R_i\} \cup \{V_i\}$ by $\{Q_i\}$.

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